MATH 207(c)–Analysis IV

1. Calculate the Fourier transform of the following functions (the residue theorem might be useful for a few cases, but not in all of them):

(a) $f(x) = \dfrac{1}{x^2 - 2x + 2}$
Hint: If you want, you can avoid lengthy computations by using the properties of the Fourier transform and the fact that, as we computed in class last week,

$$\mathcal{F}[\frac{1}{x^2+1}](a) = \sqrt{\frac{\pi}{2}}e^{-|a|}.$$

(b) $f(x) = \dfrac{x}{x^4 + 1}$.

(c) $f(x) = e^{-|x|}$ (in this case, you should get $\hat{f}(a) = \sqrt{\frac{2}{\pi}}\frac{1}{a^2+1}$; there are a few different ways to obtain this result).

2. Let $f : \mathbb{R} \to \mathbb{C}$ be a piecewise continuous function satisfying

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < +\infty \quad \text{and} \quad \int_{-\infty}^{+\infty} |x \cdot f(x)|\, dx < +\infty.$$

Show that the Fourier transform of $g(x) = x \cdot f(x)$ is well-defined and satisfies

$$\hat{g}(a) = i\frac{d}{da}\hat{f}(a).$$

Use this result to calculate the Fourier transform of $f(x) = xe^{-|x|}$.

3. Consider the following ordinary differential equation:

$$y''(x) + 2y'(x) + 5y(x) = e^{-|x|}, \quad x \in \mathbb{R}. \tag{1}$$

(a) By considering the Fourier transform of the above expression, find an expression for $\mathcal{F}[y](a)$.

(b) By considering the inverse Fourier transform of the expression for $\mathcal{F}[y](a)$, determine a solution $y(x)$ to (1).

Remark. The equation (1) is 2^{nd} order and has no initial or boundary value conditions, so one would expect to have a 2-dimensional space of solutions, not just a single solution. This is indeed true; however, among the solutions in this 2 dimensional space, only one goes to 0 as $x \to \pm\infty$; this is the only one for which the Fourier transfor $\mathcal{F}[y]$ is well-defined (since for the others the corresponding integral does not converge), and hence, this is exactly the one which is "selected" by our method above. In other words, applying the Fourier transform to (1) implicitly requires assuming that $y(x) \to 0$ as $x \to \pm\infty$ in order for the transform to be well defined, and this corresponds to imposing two boundary conditions at $x = \pm\infty$.

4. Use the various properties of the Fourier transform (i.e. about translations, re-scalings, frequency shifts etc, as well as the property proved in Ex. 2) together with the table of Fourier transforms given at the end of the sheet to calculate the Fourier transforms of the following functions:

(a) $f(x) = \dfrac{e^{ix}}{\beta^2 + \sigma^2 x^2}$, $\beta, \sigma \in \mathbb{R} \setminus \{0\}$.

(b) $g(x) = x^2 e^{-\beta^2 x^2}$, $\beta \in \mathbb{R} \setminus \{0\}$.

(c) $h(x) = \dfrac{x^2 - 2x + 1}{(x^2 - 2x + 2)^2}$.

5. Using the properties of the Laplace transform that we saw in class, show that the indicated $\gamma_0 \in \mathbb{R}$ is an abscissa of convergence and compute the Laplace transforms of the following functions $f : [0, +\infty) \to \mathbb{C}$:

(a) $f(t) = (t+1)^3$, $\gamma_0 = 0$.

(b) $f(t) = \sin(\omega t)$ (where $\omega \in \mathbb{R}$), $\gamma_0 = 0$.

(c) $f(t) = t^2 \cos(\omega t)$ (where $\omega \in \mathbb{R}$), $\gamma_0 = 0$.

(d) $f(t) = \cosh(\omega t)$ (where $\omega \in \mathbb{R}$), $\gamma_0 = |\omega|$.

6. For two piecewise continuous functions $f, g : [0, +\infty) \to \mathbb{C}$, we define their convolution $f * g : [0, +\infty) \to \mathbb{C}$ by the relation

$$f * g(t) \doteq \int_0^t f(s) g(t-s)\, ds.$$

(a) Show that the above definition coincides with the usual definition of the convolution of $f, g : \mathbb{R} \to \mathbb{C}$ if we assume that f, g are extended on $(-\infty, 0)$ by the requirement that they are identically 0 there.

(b) Show that the Laplace transform of $f * g$ satisfies

$$\mathcal{L}[f * g](z) = \mathcal{L}[f](z) \cdot \mathcal{L}[g](z)$$

for any $z \in \mathbb{C}$ for which $\mathcal{L}[f]$ and $\mathcal{L}[g]$ are well-defined *(Hint: Write down the expression for the Laplace transform and use the (trivial) identity $e^{-zt} = e^{-zs} e^{-z(t-s)}$).*

Solutions

Problem 1

Calculate the Fourier transform of the following functions:

(a) $f(x) = \frac{1}{x^2-2x+2}$

We note:

$$x^2 - 2x + 2 = (x-1)^2 + 1$$

Thus:

$$f(x) = \frac{1}{(x-1)^2+1}$$

Given that:

$$\mathcal{F}\left[\frac{1}{x^2+1}\right](a) = \sqrt{\frac{\pi}{2}}e^{-|a|}$$

Using the translation property of the Fourier transform (namely that $\mathcal{F}[f(x-x_0)](a) = e^{-iax_0}\mathcal{F}[f(x)](a)$):

$$\mathcal{F}[f(x)](a) = e^{-ia}\sqrt{\frac{\pi}{2}}e^{-|a|}$$

Answer:

$$\boxed{\hat{f}(a) = \sqrt{\frac{\pi}{2}}e^{-ia}e^{-|a|}}$$

(b) $f(x) = \frac{x}{x^4+1}$

Actually the Fourier transform of f was computed in Exercise 8.2 (I was planning to give a slightly different combination of polynomials for this exercise sheet, but I mistakenly gave the same; the method of calculation would have been the same in any case). The Fourier transform of f

(c) $f(x) = e^{-|x|}$

The Fourier transform is computed by splitting the integral:

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} e^{-iax}e^{-|x|}\,dx\right) = \frac{1}{\sqrt{2\pi}}\left(\int_0^\infty e^{(-ia-1)x}\,dx + \int_{-\infty}^0 e^{(-ia+1)x}dx\right)$$

We can easily calculate both integrals (note that the corresponding boundary terms at $x = \pm\infty$ vanish):

$$\int_0^\infty e^{-(1+ia)x}dx = \frac{1}{1+ia}, \qquad \int_{-\infty}^0 e^{(1-ia)x}dx = \frac{1}{1-ia}$$

Thus:

$$\hat{f}(a) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+ia} + \frac{1}{1-ia}\right) = \frac{1}{\sqrt{2\pi}}\frac{2}{1+a^2}$$

Answer:

$$\boxed{\hat{f}(a) = \sqrt{\frac{2}{\pi}}\frac{1}{1+a^2}}$$

Problem 2

Fourier transform of $g(x) = xf(x)$

Proof

By definition:

$$\hat{g}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} xf(x)e^{-iax}dx$$

Note:

$$xe^{-iax} = i\frac{d}{da}e^{-iax}$$

Thus:

$$\begin{aligned}
\hat{g}(a) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} xf(x)e^{-iax}dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)i\frac{d}{da}\left(e^{-iax}\right)dx \\
&= i\frac{d}{da}\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)e^{-iax}dx\right) \\
&= i\frac{d}{da}\left(\hat{f}(a)\right).
\end{aligned}$$

Application to the case of $xe^{-|x|}$

We have from Problem 1(c): If $h(x) = e^{-|x|}$,

$$\hat{h}(a) = \sqrt{\frac{2}{\pi}}\frac{1}{1+a^2}$$

Thus:

$$\frac{d}{da}\left(\frac{1}{1+a^2}\right) = \frac{-2a}{(1+a^2)^2}$$

Thus: If $g(x) = xe^{-|x|} =$ x h(x),

$$\hat{g}(a) = i\frac{d}{da}\hat{h}(a) = i\sqrt{\frac{2}{\pi}}\left(\frac{-2a}{(1+a^2)^2}\right) = -i\sqrt{\frac{8}{\pi}}\frac{a}{(1+a^2)^2}.$$

Problem 3

Solve the differential equation:

$$y''(x) + 2y'(x) + 5y(x) = e^{-|x|}$$

(a) Find $\widehat{y}(a)$

Apply the Fourier transform to each term separately. Recall:

- $\mathcal{F}[y(x)](a) = \widehat{y}(a)$
- $\mathcal{F}[y'(x)](a) = ia\widehat{y}(a)$
- $\mathcal{F}[y''(x)](a) = -a^2\widehat{y}(a)$

Thus, taking Fourier transforms term-by-term:

$$-a^2\widehat{y}(a) + 2ia\widehat{y}(a) + 5\widehat{y}(a) = \mathcal{F}[e^{-|x|}](a)$$

Since:

$$\mathcal{F}[e^{-|x|}](a) = \sqrt{\frac{2}{\pi}}\frac{1}{1+a^2},$$

we obtain:

$$\left(-a^2 + 2ia + 5\right)\widehat{y}(a) = \sqrt{\frac{2}{\pi}}\frac{1}{1+a^2}$$

Thus:

$$\boxed{\widehat{y}(a) = \sqrt{\frac{2}{\pi}}\frac{1}{(1+a^2)(-a^2+2ia+5)}}$$

(b) Find $y(x)$

Using the definition of the inverse Fourier transform, we have:

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{+iax}\widehat{y}(a)\,da = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{iax}\frac{1}{(1+a^2)(-a^2+2ia+5)}\,da. \tag{2}$$

The above is again an integral of the form $\int_{-\infty}^{+\infty}\frac{p(a)}{q(a)}e^{iax}\,da$ with $\deg(q) \geqslant \deg(p)+2$, which we have seen how to compute in our applications of the residue theorem. In particular, since

$$(1+a^2)(-a^2+2ia+5) = -(a-i)(a+i)(a-2-i)(a+2-i),$$

the poles of $\frac{1}{(1+a^2)(-a^2+2ia+5)}$ are simple and lie at $a = \pm i$, $a = \pm 2 + i$. Therefore, in order to compute the integral

$$\int_{-\infty}^{+\infty} e^{iax}\frac{1}{(1+a^2)(-a^2+2ia+5)}\,da = \lim_{R\to+\infty}\int_{-R}^{+R} e^{iax}\frac{1}{(1+a^2)(-a^2+2ia+5)}\,da,$$

we distingusih two cases:

- When $x \geqslant 0$ (in which case e^{iax} is bounded for $\mathrm{Im}(a) \geqslant 0$), we compute the integral

$$\int_{-R}^{+R} e^{iax} \frac{1}{(1+a^2)(-a^2+2ia+5)}\, da$$

by forming a closed loop using a half circle of radius R in the **upper** half plane for a. In the upper half plane, the poles of the integrand are i, $2+i$ and $-2+i$. In this case, as $R \to +\infty$, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} & e^{iax} \frac{1}{(1+a^2)(-a^2+2ia+5)}\, da \\ =& 2\pi i \Bigg(\mathrm{Res}_{z=i} \left(e^{izx} \frac{1}{(1+z^2)(-z^2+2iz+5)} \right) + \mathrm{Res}_{z=2+i} \left(e^{izx} \frac{1}{(1+z^2)(-z^2+2iz+5)} \right) \\ & + \mathrm{Res}_{z=-2+i} \left(e^{izx} \frac{1}{(1+z^2)(-z^2+2iz+5)} \right) \Bigg).\end{aligned}$$

Recall from Exercise 4 in Series 7 that, in the case of a function of the form $\frac{g(z)}{h(z)}$, where $h(z_0) \neq 0$ and $g(z)$ has a simple zero at z_0, we have

$$\mathrm{Res}_{z=z_0} \left(\frac{g(z)}{h(z)} \right) = \frac{g(z_0)}{h'(z_0)}.$$

Using the above formula, we can easily compute the residues, thus yielding:

$$\begin{aligned}\int_{-\infty}^{+\infty} & e^{iax} \frac{1}{(1+a^2)(-a^2+2ia+5)}\, da \\ &= 2\pi i \left(-\frac{1}{8} i e^{-x} + \left(-\frac{1}{32} + \frac{1}{32} i \right) e^{(-1+2i)x} + \left(\frac{1}{32} + \frac{1}{32} i \right) e^{(-1-2i)x} \right) \\ &= \frac{\pi}{8} e^{-x} \left(2 - \cos(2x) + \sin(2x) \right).\end{aligned}$$

- When $x < 0$ (in which case e^{iax} is bounded for $\mathrm{Im}(a) \leqslant 0$), we compute the integral

$$\int_{-R}^{+R} e^{iax} \frac{1}{(1+a^2)(-a^2+2ia+5)}\, da$$

by forming a closed loop using a half circle of radius R in the **lower** half plane for a.

Important remark. When closing the loop in the lower half plane, if we give our curve a positive (i.e. counterclockwise) orientation, then the part of the curve on the real line is parametrized from $+R$ to $-R$ (make a drawing to verify this). Thus, the original integral (which corresponds to a parametrization going from $-R$ to $+R$ is *minus* the result obtained from the positively oriented loop. This explains the $-$ sign below (in comparison to what we would naively expect from the residue theorem).

In the lower half plane, the only pole of the integrand is $-i$ In this case, as $R \to +\infty$, we get

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{iax} \frac{1}{(1+a^2)(-a^2+2ia+5)}\, da &= -2\pi i \left(\operatorname{Res}_{z=-i}\left(e^{izx}\frac{1}{(1+z^2)(-z^2+2iz+5)}\right)\right) \\ &= -2\pi i \left(\frac{i}{16}e^x\right) \\ &= \frac{\pi}{8}e^x.\end{aligned}$$

Combing the above and returning to (2), we get

$$y(x) = \begin{cases} \frac{1}{8}e^{-x}\left(2-\cos(2x)+\sin(2x)\right), & x \geqslant 0, \\ \frac{1}{8}e^x, & x \leqslant 0. \end{cases}$$

Problem 4

Use properties of the Fourier transform to calculate the following:

(a) $f(x) = \frac{e^{ix}}{\beta^2+\sigma^2x^2}$

Using the table of Fourier transforms, we have

$$\mathcal{F}\left[\frac{1}{x^2+\beta^2}\right](a) = \sqrt{\frac{\pi}{2}}\frac{1}{|\beta}|e^{-|\beta||x|}.$$

Thus, using the property of the Fourier transforms for rescalings, we get

$$\mathcal{F}\left[\frac{1}{\sigma^2x^2+\beta^2}\right](a) = \sqrt{\frac{\pi}{2}}\frac{1}{|\beta||\sigma|}e^{-\frac{|\beta||x|}{|\sigma|}}.$$

The multiplication by e^{ix} in physical space corresponds to a frequency shift by 1 in Fourier space. Thus:

$$\boxed{\mathcal{F}[f](a) = \mathcal{F}\left[\frac{1}{\sigma^2x^2+\beta^2}\right](a-1) = \sqrt{\frac{\pi}{2}}\frac{1}{|\sigma||\beta|}e^{-\frac{|\beta|}{|\sigma|}|a-1|}}$$

(b) $g(x) = x^2e^{-\beta^2x^2}$

We know from the table of Fourier transforms (but also from our earlier computation of the Fourier transforms of Gaussian functions):

$$\mathcal{F}[e^{-\beta^2x^2}](a) = \frac{1}{\sqrt{2}|\beta|}e^{-\frac{a^2}{4\beta^2}}$$

Using the relation between the Fourier transform and differentiation, yielding $\mathcal{F}[x^2 f(x)](a) = -\frac{d^2}{da^2}\mathcal{F}[f(x)](a)$:

$$\mathcal{F}[g](a) = -\frac{d^2}{da^2}\left(\frac{1}{\sqrt{2}|\beta|}e^{-\frac{a^2}{4\beta^2}}\right)$$

Computing derivatives:

$$\boxed{\mathcal{F}[g](a) = \frac{1}{\sqrt{2}|\beta|}\left(\frac{a^2}{4\beta^4} - \frac{1}{2\beta^2}\right)e^{-\frac{a^2}{4\beta^2}}}$$

(c) $h(x) = \frac{x^2-2x+1}{(x^2-2x+2)^2}$

Notice that:

$$x^2 - 2x + 2 = (x-1)^2 + 1 \quad \text{and} \quad x^2 - 2x + 1 = (x-1)^2$$

Thus:

$$h(x) = \frac{(x-1)^2}{((x-1)^2+1)^2}$$

Let $u = x - 1$. Then:

$$h(u+1) = \frac{u^2}{(u^2+1)^2}$$

Thus, using the relation between the Fourier transform and translations:

$$\boxed{\mathcal{F}[h(x)](a) = e^{-ia} \times \left(\text{Fourier transform of } \frac{u^2}{(u^2+1)^2}\right)}$$

For the Fourier transform of $\frac{x^2}{(x^2+1)^2}$, we can use the table of Fourier transforms attached in the end of the exercises:

$$\mathcal{F}\left[\frac{x^2}{(x^2+1)^2}\right](a) = \sqrt{\frac{\pi}{2}}\left(\frac{1}{2}e^{-|a|}(1-|a|)\right)$$

Thus:

$$\boxed{\mathcal{F}[h(x)](a) = \sqrt{\frac{\pi}{2}}e^{-ia}e^{-|a|}\left(\frac{1-|a|}{2}\right)}$$

Problem 5

Laplace transforms:

(a) $f(t) = (t+1)^3$

Expand:

$$(t+1)^3 = t^3 + 3t^2 + 3t + 1$$

As we have proved in Exercise 4 from the 8^{th} exercise sheet, if $\gamma_0 \in \mathbb{R}$ is an abscissa of convergence for $\mathcal{L}[h(t)]$, it is also an abscissa of convergence for $\mathcal{L}[t^n h(t)]$ for any $n \in \mathbb{N}$. In this case, since $\gamma_0 = 0$ is an abscissa of convergence for $\mathcal{L}[1]$, it is also an abscissa of convergence for $\mathcal{L}[f(t)] = \mathcal{L}[(t+1)^3]$; note that

$$\mathcal{L}[f](z) = \mathcal{L}[t^3](z) + 3\mathcal{L}[t^2](z) + 3\mathcal{L}[t](z) + \mathcal{L}[1](z).$$

Using:

$$\mathcal{L}[t^n](z) = \frac{n!}{z^{n+1}}, \quad \mathcal{L}[1](z) = \frac{1}{z}$$

Thus:

$$\mathcal{L}[f](z) = \frac{6}{z^4} + \frac{6}{z^3} + \frac{3}{z^2} + \frac{1}{z}$$

(b) $f(t) = \sin(\omega t)$

Since $|\sin(\omega t)| \leqslant 1$ (this holds since $\omega \in \mathbb{R}$), we expect 0 to be an abscissa of convergence for $\sin(\omega t)$ (since it is so for the constant function 1). Recall that the optimal abscissa of convergence captues the asymptotic exponential growth (or decay) rate of a function as $t \to +\infty$. Indeed, for any $\gamma > \gamma_0 = 0$, we have

$$\int_0^{+\infty} |f(t)| e^{-\gamma t}\, dt = \int_0^{+\infty} |\sin(\omega t)| e^{-\gamma t}\, dt \leqslant \int_0^{+\infty} e^{-\gamma t}\, dt = \frac{1}{\gamma} < +\infty,$$

so $\gamma_0 = 0$ is an abscissa of convergence in this case.

Using directly the definition of the Laplace transform, we calculate for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \gamma_0 = 0$:

$$\begin{aligned}
\mathcal{L}[\sin(\omega t)](z) &= \int_0^{+\infty} \sin(\omega t) e^{-zt}\, dt \\
&= \int_0^{+\infty} \frac{e^{i\omega t} - e^{-i\omega t}}{2i} e^{-zt}\, dt \\
&= \frac{1}{2i} \int_0^{+\infty} \left(e^{(-z+i\omega)t} - e^{(-z-i\omega)t} \right) dt \\
&= \frac{1}{2i} \left(\left[\frac{e^{(-z+i\omega)t}}{-z+i\omega} \right]_{t=0}^{+\infty} - \left[\frac{e^{(-z-i\omega)t}}{-z-i\omega} \right]_{t=0}^{+\infty} \right) \\
&= \frac{1}{2i} \left(-\frac{1}{-z+i\omega} + \frac{1}{-z-i\omega} \right) \\
&= \frac{\omega}{z^2+\omega^2}.
\end{aligned}$$

In the above, the upper limits in the result of the integration vanish, i.e. $\lim_{t\to} e^{(-z\pm i\omega)t} = 0$, since $\mathrm{Re}(z) > 0$.

(c) $f(t) = t^2\cos(\omega t)$

Similarly as in the case of $\sin(\omega t)$ that we calculated above, we have that $\gamma_0 = 0$ is an abscissa of converegence for $\mathcal{L}[\cos(\omega t)]$. Thus, it is also an abscissa of convergence for $\mathcal{L}[t^2\cos(\omega t)]$. Moreover, we can compute (similarly as in the case for $\sin(\omega t)$) for any $z \in \mathbb{C}$ with $\mathrm{Re}(z) > \gamma_0 = 0$:

$$\begin{aligned}
\mathcal{L}[cos(\omega t)](z) &= \int_0^{+\infty} \sin(\omega t)e^{-zt}\,dt \\
&= \int_0^{+\infty} \frac{e^{i\omega t}+e^{-i\omega t}}{2}e^{-zt}\,dt \\
&= \frac{1}{2}\int_0^{+\infty} \left(e^{(-z+i\omega)t}+e^{(-z-i\omega)t}\right)\,dt \\
&= \frac{1}{2}\left(\left[\frac{e^{(-z+i\omega)t}}{-z+i\omega}\right]_{t=0}^{+\infty} + \left[\frac{e^{(-z-i\omega)t}}{-z-i\omega}\right]_{t=0}^{+\infty}\right) \\
&= \frac{1}{2}\left(-\frac{1}{-z+i\omega} - \frac{1}{-z-i\omega}\right) \\
&= \frac{z}{z^2+\omega^2}.
\end{aligned}$$

Therefore

$$\mathcal{L}[t^2\cos(\omega t)] = \left(-\frac{d}{dz}\right)^2 \mathcal{L}[\cos(\omega t)](z) = \frac{2(z^2-\omega^2)}{(z^2+\omega^2)^3}.$$

(d) $f(t) = \cosh(\omega t)$

Recall:

$$\cosh(\omega t) = \frac{e^{\omega t}+e^{-\omega t}}{2}$$

As we have seen in class, for any $a \in \mathbb{R}$, $\gamma_0 = a$ is an abscissa of convergence for $\mathcal{L}[e^{at}](z)$ and we have

$$\mathcal{L}[e^{at}](z) = \frac{1}{z-a}.$$

Thus, $\gamma_0 = |\omega|$ (namely the largest between $+\omega$ and $-\omega$) is an abscissa of convergence for $\cosh(\omega t)$ and:

$$\mathcal{L}[\cosh(\omega t)](z) = \frac{1}{2}\left(\frac{1}{z-\omega}+\frac{1}{z+\omega}\right) = \frac{z}{z^2-\omega^2} \quad (\mathrm{Re}(z) > |\omega|).$$

Problem 6

Convolution and Laplace transforms:

(a) Extension to $\mathbb{R}$

Extending $f(t)$ and $g(t)$ to be zero for $t < 0$, we have that $f(s) = 0$ for $s < 0$ and $g(t-s) = 0$ for $s > t$. Therefore, $f(s) \cdot g(t-s) = 0$ for $s \notin [0, t]$. Thus,

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(s)g(t-s)\,ds = \int_0^t f(s)g(t-s)\,ds.$$

(b) Laplace transform of convolution

We compute

$$\begin{aligned}
\mathcal{L}[f*g](z) &= \int_0^{+\infty} (f*g)(t)e^{-zt}\,dt \\
&= \int_0^{+\infty} \left(\int_0^t f(s)g(t-s)\,ds \right) e^{-zt}\,dt \\
&= \int_0^{+\infty} \int_0^t f(s)g(t-s)e^{-zt}\,dsdt \\
&= \int_0^{+\infty} \int_0^t \left(f(s)e^{-zs}\right) \left(g(t-s)e^{-z(t-s)}\right)\,dsdt.
\end{aligned}$$

We can now switch the order of integration: Noting that the region in the (s,t) plane over which we are integrating is $\{0 \leqslant s \leqslant t\} \cap \{0 \leqslant t < +\infty\}$ (i.e. the region between $s = 0$ axis and the diagonal $s = t$), which can be equivalently expressed as $\{t \geqslant s\} \cap \{0 \leqslant s < +\infty\}$, we obtain that the above integral is equal to

$$\int_0^{+\infty} \int_s^{+\infty} \left(f(s)e^{-zs}\right) \left(g(t-s)e^{-z(t-s)}\right)\,dtds.$$

Setting $u = t - s$, the above integral becomes

$$\int_0^{+\infty} \int_0^{+\infty} \left(f(s)e^{-zs}\right) \left(g(u)e^{-zu}\right)\,duds = \left(\int_0^{+\infty} f(s)e^{-zs}\,ds \right) \cdot \left(\int_0^{+\infty} g(u)e^{-zu}\,du \right) = \mathcal{L}[f](z) \cdot \mathcal{L}[g](s),$$

which proves the required result.